F) ELEMENTS OF THIN PLATES THEORY

F.1) The model of a thin elastic plane plate.

The thin plane plate is a cylindrical body having an arbitrary horizontal cross section in the initially non-deformed state. Its height, denoted by $H = 2h$, is much smaller than the other dimensions (usually, around 7-10 times). The material of the plate is an elastic, homogenous, isotropic one, having the constants denoted by E and ν, respectively λ and μ. The third axis of the co-ordinate system, denoted by Oz, is a vertical one, positive downward. Let be $x_1 = x$, $x_2 = y$. In the initial, non-deformed state, the plate is a plane horizontal one. The upper face has the equation $z = -h$ and the down face has $z = h$ respectively. The plane of equation $z = 0$ represents the *median plane*. After the plate is deformed, it becomes a *median surface*.

F.2) The planar state of a plate. The bending state.

Two particular situation for a deformed plate are considered (Fig.F1).

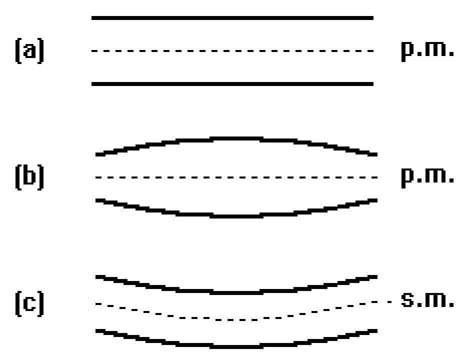

Fig.F1. (a) The non-deformed plate; (b) The plate into a planar state; (c) The plate into a bending state. Here, the median plane is denoted by **m.p.** while **m.s.** is the median surface.

In the first case, the horizontal components of the displacement vector are symmetrical ones with respect to the median plane, being even functions with respect to the z-variable, while the vertical component of the displacement vector is an anti-symmetrical one (odd function with respect to z), i.e.

$$\begin{aligned} &u_k(x, y, -z) = u_k(x, y, z), k = 1,2 \\ &u_3(x, y, -z) = -u_3(x, y, -z). \end{aligned} \tag{f1}$$

From eq.(f1b) it follows that

$$u_3(x, y, 0) = 0 \quad , \tag{f2}$$

i.e. the material points placed initially into the median plane have no vertical displacement as a consequence of the deformation of the plate (the median plane holds a horizontal plane one). This kind of deformation represents the *planar state* of the plate.
In the second case, the horizontal components of the displacement vector are anti-symmetrical ones (odd functions with respect to z), while the vertical component of the displacement vector is a symmetrical one with respect to the median plane (even function with respect to z), i.e.

$$\begin{aligned} &u_k(x, y, -z) = -u_k(x, y, z), k = 1,2 \\ &u_3(x, y, -z) = u_3(x, y, -z). \end{aligned} \tag{f3}$$

By deformation, the material points initially placed into the median plane are displaced on the vertical direction, having no horizontal movement. The median plane becomes a median surface. This state represents the *bending state* of a plate.

F.3) Loads acting on the plate.

For simplicity, the forces acting on the lateral surface of the plate are neglected. On the upper surface of the plate, having the equation $z=-h$ and the outward pointing normal vector $\vec{n} = -\vec{e}_3$, it is acting the surface force $\vec{q}^{\,s} = \vec{q}^{\,s}(x,y)$. In the same way, on the down surface, having the equation $z=h$ and the normal vector $\vec{n} = \vec{e}_3$, it is acting the surface force $\vec{q}^{\,j} = \vec{q}^{\,j}(x,y)$. Hence

$$\sigma\left(-\vec{e}_3\right) = \vec{q}^{\,s} \text{ , for } z=-h \quad , \tag{f4}$$

i.e.

$$\sigma_{13}(x,y,-h) = -q_1^s(x,y), \sigma_{23}(x,y,-h) = -q_2^s(x,y), \sigma_{33}(x,y,-h) = -q_3^s(x,y) \tag{f5}$$

Also

$$\sigma\,\vec{e}_3 = \vec{q}^{\,j} \text{ , for } z=h \quad , \tag{f6}$$

i.e.

$$\sigma_{13}(x,y,h) = q_1^j(x,y), \sigma_{23}(x,y,h) = q_2^j(x,y), \sigma_{33}(x,y,h) = q_3^j(x,y) . \tag{f7}$$

We shell see that the above presented deformation states are compatible only to certain distributions of volume or surface forces applied to the plate.

F.4) Odd and even functions for the planar state and for the bending state.

Let $f = f(x,y,z)$ be a function of three variables, supposed to be smooth enough. It can be seen that

$$f(x,y,z) = \frac{f(x,y,z)+f(x,y,-z)}{2} + \frac{f(x,y,z)-f(x,y,-z)}{2} \tag{f8}$$

Let

$$f^+(x,y,z) = \frac{f(x,y,z)+f(x,y,-z)}{2} \quad , \tag{f9}$$

$$f^-(x,y,z) = \frac{f(x,y,z)-f(x,y,-z)}{2} \tag{f10}$$

It follows that

$$f^+(x,y,-z) = f^+(x,y,z) \quad , \quad f^-(x,y,-z) = -f^-(x,y,z) \quad . \tag{f11}$$

The function f^+ represents the even part of f (with respect to z-variable), while the function f^- represents its odd part. It follows that

$$\left(f^+\right)^+ = f^+ \quad , \quad \left(f^-\right)^- = f^- \quad , \quad \left(f^+\right)^- = \left(f^-\right)^+ = 0 \quad , \tag{f12}$$

i.e. the even part of the even part is equal to the even part too. A similar relation holds for the odd part. The even part of an odd part (and the odd part of an even part) is vanishing. For $k=1,2 \quad , \quad x_1 = x \quad , \quad x_2 = y$, it follows that

$$\left(f^+\right)_{,k} = \frac{\partial}{\partial x_k} f^+(x,y,z) = \frac{1}{2}\left[\frac{\partial}{\partial x_k} f(x,y,z) + \frac{\partial}{\partial x_k} f(x,y,-z)\right] = \left[f_{,k}(x,y,z)\right]^+ \tag{f13}$$

Hence, the partial derivative (with respect to a horizontal co-ordinate) of the even part of a certain function is equal to the even part of that derivative. This property holds for the odd part. So,

$$\left(f^+\right)_{,k} = \left(f_{,k}\right)^+ \quad , \quad \left(f^-\right)_{,k} = \left(f_{,k}\right)^- \tag{f14}$$

Also,

$$\left(f^{+}\right)_{,3}=\frac{1}{2}\left\{\frac{\partial}{\partial z}\left[f(x,y,z)+f(x,y,-z)\right]\right\}=\frac{1}{2}\left[f_{,3}(x,y,z)-f_{,3}(x,y,-z)\right]=\left(f_{,3}\right)^{-} \quad \text{(f15)}$$

Hence the partial derivative of the even part with respect to the vertical co-ordinate is equal to the odd part of the partial derivative of the function itself with respect to same co-ordinate. In the same way it follows that

$$\left(f^{-}\right)_{,3}=\left(f_{,3}\right)^{+} \quad . \quad \text{(f16)}$$

By using the HOOKE's law and the definition of the strain, it can be resumed that the planar state and the bending state are characterised by the next components:

-for the *planar state*: - the displacement vector: $\overset{+}{u}_1, \overset{+}{u}_2, \overset{-}{u}_3$,

- strain tensor: $\overset{+}{\varepsilon}_{11}, \overset{+}{\varepsilon}_{12}, \overset{-}{\varepsilon}_{13}, \overset{+}{\varepsilon}_{22}, \overset{-}{\varepsilon}_{23}, \overset{+}{\varepsilon}_{33}, \vartheta^{+}$,

- stress tensor: $\overset{+}{\sigma}_{11}, \overset{+}{\sigma}_{12}, \overset{-}{\sigma}_{13}, \overset{+}{\sigma}_{22}, \overset{-}{\sigma}_{23}, \overset{+}{\sigma}_{33}$.

-for the *bending state*: - the displacement vector: $\overset{-}{u}_1, \overset{-}{u}_2, \overset{+}{u}_3$,

- strain tensor: $\overset{-}{\varepsilon}_{11}, \overset{-}{\varepsilon}_{12}, \overset{+}{\varepsilon}_{13}, \overset{-}{\varepsilon}_{22}, \overset{+}{\varepsilon}_{23}, \overset{-}{\varepsilon}_{33}, \vartheta^{-}$,

- stress tensor: $\overset{-}{\sigma}_{11}, \overset{-}{\sigma}_{12}, \overset{+}{\sigma}_{13}, \overset{-}{\sigma}_{22}, \overset{+}{\sigma}_{23}, \overset{-}{\sigma}_{33}$.

F5) Mean value of a function. Equilibrium equations for thin plates.

Let $f=f(x,y,z)$ an integrable function with respect to z-variable. The mean value of f computed on the thickness of the plate is denoted by

$$\bar{f}(x,y)=\frac{1}{2h}\int_{-h}^{+h} f(x,y,z)dz \quad . \quad \text{(f17)}$$

For an odd function f, its mean value vanishes, i.e.

$$\overline{f^{-}}(x,y)=0 \quad , \quad \text{(f18)}$$

It follows that

$$\bar{f}=\overline{f^{+}+f^{-}}=\overline{f^{+}}+\overline{f^{-}}=\overline{f^{+}} \quad \text{(f19)}$$

For a function $C=C(x,y)$ depending only on the horizontal co-ordinates, eq.(f17) leads to

$$\bar{C}=C \quad , \quad \overline{zC}=0 \quad . \quad \text{(f20)}$$

Differentiating eq.(f17) with respect to the horizontal co-ordinates, it follows that:

$$\overline{f_{,k}}=\left(\bar{f}\right)_{,k} \quad , \quad k=1,2 \quad . \quad \text{(f21)}$$

For the vertical derivative, it follows that

$$\overline{f_{,3}}=\frac{1}{2h}\int_{-h}^{+h}\frac{\partial f}{\partial z}dz=\frac{1}{2h}\left[f(x,y,h)-f(x,y,-h)\right]=\frac{1}{h}f^{-}(x,y,z=h) \quad . \quad \text{(f22)}$$

Consider the function $zf(x,y,z)$. It follows that

$$zf^{+}=(zf)^{-} \quad , \quad zf^{-}=(zf)^{+} \quad . \quad \text{(f23)}$$

Its mean value is

$$\overline{zf}=\overline{z(f^{+}+f^{-})}=\overline{zf^{+}}+\overline{zf^{-}}=\overline{zf^{-}}=\overline{(zf)^{+}} \quad . \quad \text{(f24)}$$

Hence

$$\overline{zf_{,3}} = \frac{1}{2h}\int_{-h}^{+h} z\frac{\partial f}{\partial z}dz = \frac{1}{2h}\int_{-h}^{+h}\left[\frac{\partial}{\partial z}(zf) - f\right]dz = \frac{1}{2h}\left[(zf)_{-h}^{+h}\right] - \frac{1}{2h}\int_{-h}^{+h} f dz = \frac{f(x,y,h)+f(x,y,-h)}{2} - \bar{f}(x,y) = f^{+}(x,y,z=h) - \bar{f} \quad . \qquad \text{(f25)}$$

Neglecting the volume forces (the weight of the plate itself, for example), the equilibrium equations for the planar state are

$$\begin{aligned} &\left(\sigma_{11}^{+}\right)_{,1} + \left(\sigma_{12}^{+}\right)_{,2} + \left(\sigma_{13}^{-}\right)_{,3} = 0 \\ &\left(\sigma_{12}^{+}\right)_{,1} + \left(\sigma_{22}^{+}\right)_{,2} + \left(\sigma_{23}^{-}\right)_{,3} = 0 \, . \\ &\left(\sigma_{13}^{-}\right)_{,1} + \left(\sigma_{23}^{-}\right)_{,2} + \left(\sigma_{33}^{+}\right)_{,3} = 0 \end{aligned} \qquad \text{(f26)}$$

Let the mean values of the stress components be denoted by

$$\Sigma_{ij} = \overline{\sigma_{ij}} = \frac{1}{2h}\int_{-h}^{+h}\sigma_{ij}(x,y,z)dz \quad i,j=1,2,3 \quad , \qquad \text{(f27)}$$

Applying the mean value operator to eqs. (f26) gives

$$\begin{aligned} &(\Sigma_{11})_{,1} + (\Sigma_{12})_{,2} + \frac{1}{2h}\left[\sigma_{13}(x,y,h) - \sigma_{13}(x,y,-h)\right] = 0 \\ &(\Sigma_{12})_{,1} + (\Sigma_{22})_{,2} + \frac{1}{2h}\left[\sigma_{23}(x,y,h) - \sigma_{23}(x,y,-h)\right] = 0 \end{aligned} \quad , \qquad \text{(f28)}$$

Using eqs.(f5) and (f7) it follows

$$\begin{aligned} &(\Sigma_{11})_{,1} + (\Sigma_{12})_{,2} + \frac{1}{2h}\left[q_1^{j}(x,y) + q_1^{s}(x,y)\right] = 0 \\ &(\Sigma_{12})_{,1} + (\Sigma_{22})_{,2} + \frac{1}{2h}\left[q_2^{j}(x,y) + q_2^{s}(x,y)\right] = 0 \end{aligned} \quad . \qquad \text{(f29)}$$

Let

$$M_{ij} = \overline{z\sigma_{ij}} = \frac{1}{2h}\int_{-h}^{+h} z\sigma_{ij}(x,y,z)dz \quad i,j=1,2,3 \quad . \qquad \text{(f30)}$$

Multiplying eqs.(f26) by z and using again the mean value operator, it follows

$$(M_{13})_{,1} + (M_{23})_{,2} + \frac{\sigma_{33}(x,y,h) + \sigma_{33}(x,y,-h)}{2} - \Sigma_{33} = 0 \, . \qquad \text{(f31)}$$

Hence, using again eqs.(f5) and (f7),

$$(M_{13})_{,1} + (M_{23})_{,2} + \frac{q_3^{j}(x,y) - q_3^{j}(x,y)}{2} - \Sigma_{33} = 0 \, , \qquad \text{(f32)}$$

So, the equilibrium of a thin plate into the *planar state* leads to eqs.(f29) and (f32).
Consider the weight of the plate, the equation of equilibrium for the plate into the *bending state* are

$$\begin{aligned} &\left(\sigma_{11}^{-}\right)_{,1} + \left(\sigma_{12}^{-}\right)_{,2} + \left(\sigma_{13}^{+}\right)_{,3} = 0 \\ &\left(\sigma_{12}^{-}\right)_{,1} + \left(\sigma_{22}^{-}\right)_{,2} + \left(\sigma_{23}^{+}\right)_{,3} = 0 \\ &\left(\sigma_{13}^{+}\right)_{,1} + \left(\sigma_{23}^{+}\right)_{,2} + \left(\sigma_{33}^{-}\right)_{,3} + \rho g = 0 \end{aligned} \quad . \qquad \text{(f33)}$$

Here, the density of the plate is ρ and the acceleration of gravity is g.
Proceeding in a similar manner to above, it follows the equations of equilibrium for the thin plate into a *bending state* are

$$\left(\Sigma_{13}\right)_{,1}+\left(\Sigma_{23}\right)_{,2}+\frac{1}{2h}\left[q_3^j(x,y)+q_3^s(x,y)\right]+\rho g=0,$$
$$\left(M_{11}\right)_{,1}+\left(M_{12}\right)_{,2}+\frac{1}{2}\left[q_1^j(x,y)-q_1^s(x,y)\right]-\Sigma_{13}=0 \quad . \qquad \text{(f34)}$$
$$\left(M_{12}\right)_{,1}+\left(M_{22}\right)_{,2}+\frac{1}{2}\left[q_2^j(x,y)-q_2^s(x,y)\right]-\Sigma_{23}=0$$

Eq.(f34b) is differentiated with respect to x and eq.(f34c) is differentiated with respect to y. The results are substituted in eq.(f34a). Hence

$$\begin{aligned}&\left(M_{11}\right)_{,11}+2\left(M_{12}\right)_{,12}+\left(M_{22}\right)_{,22}+\frac{1}{2h}\left[q_3^j(x,y)+q_3^s(x,y)\right]+\rho g\\&+\frac{1}{2}\left[q_1^j(x,y)-q_1^s(x,y)\right]_{,1}+\frac{1}{2}\left[q_2^j(x,y)-q_2^s(x,y)\right]_{,2}=0\end{aligned} \quad . \qquad \text{(f35)}$$

F.6) Thin plane plate in the bending state.

The component u_1 of the displacement vector is developed in power series with respect to z-variable. It follows

$$u_1(x,y,z)=u_1(x,y,0)+z\frac{\partial u_1}{\partial z}(x,y,0)+\ldots \qquad \text{(f36)}$$

For the bending state, the first term in eq.(f36) vanishes. Because the thickness of the plate is a small one, only the second term is kept. So the horizontal components of the displacement vector are

$$u_1(x,y,z)=z\frac{\partial u_1}{\partial z}(x,y,0) \quad , \quad u_2(x,y,z)=z\frac{\partial u_2}{\partial z}(x,y,0) \quad . \qquad \text{(f37)}$$

The vertical displacement of the points placed into the median plane is denoted by

$$w=w(x,y)=u_3(x,y,0) \quad , \qquad \text{(f38)}$$

i.e. the mean surface has the equation $z=w(x,y)$. Here, w represents *the arrow of the plate*.

F7) BERNOULLI's hypothesis.

According to BERNOULLI, it is assumed that an arbitrary material segment of the plate, initially perpendicular on the median plane in the non-deformed state, rests perpendicular on the mean surface in the deformed state. Let $A(x,y,z)$ a certain point of the plate (not placed in the mean plane) and $A_0(x,y,0)$ its projection on the median plane. So, the segment $\overrightarrow{A_0A}$ is perpendicular on the median plane. After deformation, the material point A is moving at the point having the co-ordinates $M\left(x+u_1(x,y,z),y+u_2(x,y,z),z+u_3(x,y,z)\right)$, while the point A_0 is moving at the point $M_0\left(x+u_1(x,y,0),y+u_2(x,y,0),w(x,y)\right)$. The horizontal displacements of the points placed in the median plane are vanishing. So, writing the vector along the line, it follows that

$$\overrightarrow{M_0M}=\left(u_1(x,y,z),\,u_2(x,y,z),\,z+u_3(x,y,z)-w(x,y)\right) \qquad \text{(f39)}$$

Expanding in power series the even function u_3 with respect to z-variable, it follows

$$u_3(x,y,z)=w(x,y)+z^2\beta(x,y)+\ldots \qquad \text{(f40)}$$

Using eqs.(f37) and (f40), it follows that

$$\overrightarrow{M_0M} = \left(z\frac{\partial u_1}{\partial z}(x,y,0),\, z\frac{\partial u_2}{\partial z}(x,y,0),\, z + z^3\beta(x,y)+\ldots \right)$$
$$= z\left(\frac{\partial u_1}{\partial z}(x,y,0),\, \frac{\partial u_2}{\partial z}(x,y,0),\, 1 + z^2\beta(x,y)+\ldots \right) \quad \text{(f41)}$$

Hence the direction is

$$\frac{\overrightarrow{M_0M}}{||\overrightarrow{M_0M}||} = \frac{\left(\frac{\partial u_1}{\partial z}(x,y,0),\, \frac{\partial u_2}{\partial z}(x,y,0),\, 1 + z^2\beta(x,y)+\ldots \right)}{\sqrt{\left(\frac{\partial u_1}{\partial z}\right)^2 + \left(\frac{\partial u_2}{\partial z}\right)^2 + \left(1 + z^2\beta(x,y)+\ldots\right)^2}} \quad \text{(f42)}$$

The normal vector on the median surface is

$$\vec{n} = \left(-\frac{\partial w}{\partial x}, -\frac{\partial w}{\partial y}, 1 \right) \Big/ \sqrt{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + 1} \ . \quad \text{(f43)}$$

Comparing eq.(f42) to eq.(f43), the supplemental hypothesis of BERNOULLI is satisfied if

$$\frac{\partial u_1}{\partial z}(x,y,0) = -\frac{\partial w}{\partial x} \quad , \quad \frac{\partial u_2}{\partial z}(x,y,0) = -\frac{\partial w}{\partial y} \quad , \quad u_3(x,y,z) = w(x,y) \quad , \quad \text{(f44)}$$

i.e., by using eq.(f37), the displacement vector has the horizontal components equal to

$$u_1(x,y,z) = -z\frac{\partial w}{\partial x} \quad , \quad u_2(x,y,z) = -z\frac{\partial w}{\partial y} \quad , \quad u_3(x,y,z) = w(x,y) \quad . \quad \text{(f45)}$$

From eq. (f45), the components of the strain tensor are

$$\varepsilon_{11} = -z\frac{\partial^2 w}{\partial x^2} \quad , \varepsilon_{12} = -z\frac{\partial^2 w}{\partial x \partial y} \quad , \varepsilon_{22} = -z\frac{\partial^2 w}{\partial y^2} \quad , \varepsilon_{33} = 0 \quad . \quad \text{(f46)}$$

The LAPLACE operator in horizontal components is denoted by

$$\Delta^* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad . \quad \text{(f47)}$$

Hence the trace of the strain tensor is

$$\theta = \mathrm{tr}\,\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = -z\Delta^* w \quad . \quad \text{(f48)}$$

F8) HOOKE’s law for a thin plate.

By using eqs.(f46) and (f48) it follows that

$$\sigma_{11} = \lambda\,\mathrm{tr}\,\varepsilon + 2\mu\,\varepsilon_{11} = -z\left(\lambda\Delta^* w + 2\mu\frac{\partial^2 w}{\partial x^2} \right) \quad . \quad \text{(f49)}$$

But

$$\overline{z^2} = \frac{1}{2h}\int_{-h}^{+h} z^2 dz = \frac{h^2}{3} \quad \text{(f50)}$$

Hence

$$M_{11} = \overline{z\,\sigma_{11}} = -\frac{h^2}{3}\left(\lambda\Delta^* w + 2\mu\frac{\partial^2 w}{\partial x^2} \right) = -\frac{H^2}{12}\left(\lambda\Delta^* w + 2\mu\frac{\partial^2 w}{\partial x^2} \right) \quad . \quad \text{(f51)}$$

In the same way,

$$M_{12} = \overline{z\sigma_{12}} = -2\mu \frac{h^2}{3} \frac{\partial^2 w}{\partial x \partial y} = -\mu \frac{H^2}{6} \frac{\partial^2 w}{\partial x \partial y} \quad , \tag{f52}$$

$$M_{22} = \overline{z\sigma_{22}} = -\frac{h^2}{3}\left(\lambda\Delta^* w + 2\mu \frac{\partial^2 w}{\partial y^2}\right) = -\frac{H^2}{12}\left(\lambda\Delta^* w + 2\mu \frac{\partial^2 w}{\partial y^2}\right) \tag{f53}$$

Substituting eqs.(f51)-(f53) into eq.(f35) it follows the *equation of Sophie GERMAIN*:

$$D\Delta^*\Delta^* w = \rho g H + q_3^j(x,y) + q_3^s(x,y) + \frac{H}{2}\left\{\left[q_1^j(x,y) - q_1^s(x,y)\right]_{,1} + \left[q_2^j(x,y) - q_2^s(x,y)\right]_{,2}\right\} \quad , \tag{f54}$$

where

$$D = (\lambda + 2\mu)\frac{2h^3}{3} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{H^3}{12} \tag{f55}$$

represents the flexural *rigidity of the plate*.

So, obtaining the median surface asks someone to solve a bi-harmonic equation, with certain boundary conditions on the upper/ lower faces of the plate. The equation (f55) will be solved in same cases of particular importance in real cases.

EXERCISE. Find the expressions of Σ_{13} and Σ_{23} for a thin plate into a bending state.

F.9) The infinite, 1-dimensional (1-D) plate. The flexure of the lithosphere.

Consider an infinite extended plate along y-co-ordinate. The component u_2 of the displacement vector is equal to zero, and the rest of the components does not depend on y-co-ordinate. In this case, the plate is assumed to be in a cylindrical bending state. Neglecting the horizontal loads, eq.(f55) becomes

$$D\frac{d^4 w}{dx^4} = \rho g H + q_3^j(x) + q_3^s(x) \quad . \tag{f56}$$

For the case presented in Fig.F2, on the upper face of the plate is acting the load P due to the relief and the lithostatic pressure, i.e.

$$q_3^s(x) = P + \rho_r g(w(x) - h) \quad , \tag{f57}$$

where ρ_r is the density of the filling sediments (assumed to be homogeneous ones) placed between the reference plane of elevation equal to zero and the upper surface of the plate. On the down face of the plate, it is acting the pressure of the liquid of density equal to ρ_m ,i.e.

$$q_3^j(x) = p_0 - \rho_m g(w(x) + h) \quad , \tag{f58}$$

where p_o is an unknown constant. Eq.(f59) becomes

$$D\frac{d^4 w}{dx^4} = P - (\rho_m - \rho_r) g w(x) + p_0 - \rho_r g h - \rho_m g h + \rho g H \quad . \tag{f59}$$

It is assumed that in the absence of the relief (i.e. P=0), the non-deformed plate (i.e. $w(x) \equiv 0$) is in an equilibrium state under the action of its own weight and of the pressure of the liquid, i.e.

$$p_0 - \rho_r g h - \rho_m g h + \rho g H = 0 \tag{f60}$$

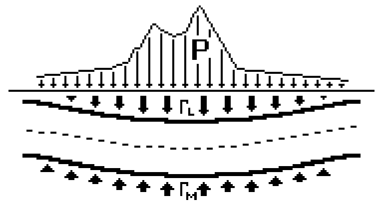

Fig.F2. The flexure of the lithosphere under the action of the relief (P), of the lithostatic pressure Γ_L and of the pressure of the liquid Γ_M .

It results the *flexure equation of the 1-D plate* :

$$\frac{d^4 w}{d x^4} = \frac{1}{D}\left[P - (\rho_m - \rho_r) g\, w(x)\right] \tag{f61}$$

Let

$$(\rho_m - \rho_r) g / D = 4 / \alpha^4 \ , \tag{f62}$$

where α is the flexural *parameter* of the plate. Eq.(f61) becomes

$$\frac{d^4 w(x)}{d x^4} = \frac{P}{D} - \frac{4}{\alpha^4} w(x) \tag{f63}$$

F.10) Exterior forces on the lateral surface of the plate. Buckling.

To derive Sophie GERMAIN equation (f56) for the bending state, exterior forces acting on the lateral surface of the plate have been ignored, especially those placed into the median plane. Consider a very thin plate simply leaning like in Fig.F3. The load P is absent and the plate is infinite developed in a direction perpendicular on the plane of the figure. An element of the plate having the length equal to unit along that direction is considered too. Let h be the thickness of the plate. The forces per unit length along the above direction are denoted by $\pm \vec{N}$, being derived from the stress σ_c (positive for compression) by

$$N = \sigma_c h \tag{f64}$$

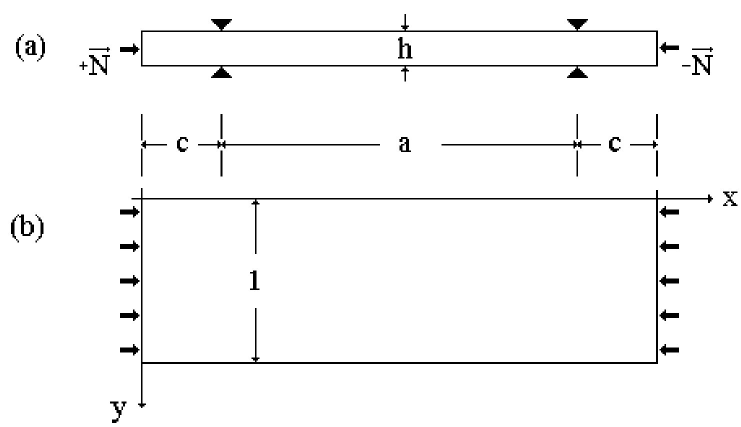

Fig.F3. A plate simply leaning, subject to the action of forces placed into the median plane.
(a)A lateral view. (b) A view from above.

If the forces $\pm \vec{N}$ are small ones, the plate will be deformed according to a plane state, attempting a final configuration similar to Fig.1b. If the forces $\pm \vec{N}$ are above a certain critical value, the plate loses suddenly its equilibrium state, attending a deformation state like Fig.F4a, (or in the contrary sense, i.e. symmetrically with respect to the line of its supports. The displacement field in this case is similar to the cylindrical bending. That phenomenon represents the buckling of the plate. It characterises very thin plates or bars.